A CLASS OF SPACES IN WHICH COMPACT SETS ARE FINITE

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INTRODUCTION

A mathematical note entitled "A Class Of Spaces In Which Compact Sets Are Finite" by Murry R. Kirch appeared in the January 1969 issue of the American Mathematical Monthly. The term of space has been introduced by Norman Levine to describe a topological space in which each compact set is finite. The term pseudo-finite has also been used to describe such spaces and will be used in this paper. Levine has shown that first countable T_1 spaces which are pseudo-finite must be discrete. Kirch shows that there exists a wide variety of nondiscrete pseudo-finite spaces. The purpose of this paper is to explain and provide proofs for all theorems and lemmas found in Kirch's paper. Examples are also provided.

Preliminary information on topological spaces is found in Chapter I. Pseudo-finite spaces have no isolated points. Some results of spaces with no isolated points are presented in Chapter II. The main results of this paper are found in Chapter III.

CHAPTER I

TOPOLOGICAL PRELIMINARIES

A topological background is needed to introduce the main results of this paper. The necessary definitions, lemmas, and theorems are presented in this chapter.

Definition 1. A topological space (X,T) is a set X of points and a family T of subsets of X which satisfy the following axioms:

1) The empty set $\emptyset$ and the whole space X are members of T.
2) The union of any number of members of T is a member of T.
3) The intersection of any finite number of members of T is a member of T.

The family T is called a topology for X, and the members of T are said to be open sets of the topological space (X,T).

Definition 2. If X^* is a subset of a topological space (X,T), the induced or relative topology for X^* is the collection T^* of all sets which are intersections of X^* with members of T. (X^*,T^*) will be called a subspace of (X,T) if and only if T^* is the relative topology.

Definition 3. Let X be any set. The family of all subsets of X yields a topology for X which is called the discrete topology for X, and X together with its discrete topology is called a discrete topological space or simply a discrete space.

Definition 4. A subset A of a discrete topological space is discrete if and only if $\{x\}$ is open relative to A for each $x \in A$.

<u>Definition 5</u>. Let X be any set. The family consisting of $\emptyset$ and X yields a topology for X called the indiscrete (or trivial) topology, and X together with its indiscrete topology is called an <u>indiscrete topological space</u> or simply an indiscrete space.

<u>Definition 6</u>. A point $x \in A$ is an <u>interior point</u> of A if and only if x belongs to some open set S_x which is contained in A; i.e., $x \in S \subset A$ where S_x is open.

<u>Definition 7</u>. A set A of a topological space X is <u>open</u> if and only if each of its points is an interior point.

<u>Definition 8</u>. Given a topological space (X,T), a point $x \in X$ is a <u>limit point</u> of a subset A of X if and only if every open set G containing x contains a point of A different from x; i.e., $x \in G \in T$, then $(A \cap G) \setminus \{x\} \neq \emptyset$. The set of all limit points of a set A is called the derived set of A and is denoted by A'.

<u>Lemma 1.1</u>. If a set A is a subset of a set B, then every limit point of A is also a limit point of B; i.e., $A \subset B \Rightarrow A' \subset B'$.

<u>Proof</u>. By definition $x \in A'$ if and only if $(G \setminus \{x\}) \cap A \neq \emptyset$ for every open set G containing x. But $B \supset A$; therefore $G \setminus \{x\} \cap B \supset G \setminus \{x\} \cap A \neq \emptyset$. Thus $x \in A' \Rightarrow x \in B'$; i.e., $A' \subset B'$.

<u>Definition 9</u>. A subset A of a topological space X is a <u>closed set</u> if and only if its complement, A^c, is an open set.

<u>Definition 10</u>. The <u>closure</u> of a set A is the intersection of all closed sets containing A. In other words, if $\{C_\alpha : \alpha \in I\}$ is the class of all closed subsets of a topological space X which contain A, then $\bar{A} = \bigcap_\alpha C_\alpha$.

<u>Proposition 1.2</u>. Let $\bar{A}$ be the closure of a set A. Then the following hold: 1) $\bar{A}$ is a closed set since A is the intersection of closed sets, 2) if C is a closed set containing A, then $A \subset \bar{A} \subset C$, and 3) A is closed if and only if $A = \bar{A}$.[1]

<u>Lemma 1.3</u>. Let A be a subset of a topological space X. Then the closure of A is the union of A and its set of limit points; i.e., $\bar{A} = A \cup A'$.[2]

<u>Lemma 1.4</u>. If a set A is a subset of a set B, then the closure of A is a subset of the closure of B; i.e., if $A \subset B$ then $\bar{A} \subset \bar{B}$.

<u>Proof</u>. If $A \subset B$, then by Lemma 1.1 $A' \subset B'$. Hence $A \cup A' \subset B \cup B'$ or by Lemma 1.3 $\bar{A} \subset \bar{B}$.

<u>Lemma 1.5</u>. Given a topological space X. A set A has no limit points if and only if it is closed and discrete.

<u>Proof</u>. Suppose A has no limit points; i.e., $A' = \emptyset$. $\bar{A} = A \cup A'$ by Lemma 1.3. So $\bar{A} = A \cup A' = A \cup \emptyset = A$. Therefore A is closed. Let $x \in A$. x is not in A' since A' is empty. Therefore there exists an

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1 H. L. Royden, <u>Real Analysis</u>, New York: Macmillan Company, 1968.

2 <u>Ibid</u>.

open set G containing x and not containing any points of A other than x. Hence $x \in G$, G is open and $G \cap (A - \{x\}) = \emptyset$. $G \cap A = \{x\}$ which is open relative to A by definition. Therefore $\{x\}$ is open in A. Thus A has the discrete topology.

Now let A be closed and discrete. Suppose $A' \neq \emptyset$. Let $x \in A'$. Since A is closed, $A = \bar{A} = A \cup A'$. Therefore $x \in A$. A is discrete, so $\{x\}$ is open in A. There exists an open set G such that $A \cap G = \{x\}$. Since $x \in A'$, G must contain a point of A other than x. This is a contradiction of the fact that $A \cap G = \{x\}$. Hence the assumption $A' \neq \emptyset$ is false. Therefore $A' = \emptyset$.

Definition 11. A subset A of a topological space X is said to be dense in $B \subset X$ if B is contained in the closure of A; i.e., $B \subset \bar{A}$. In particular, A is dense in X or is a dense subset of X if and only if $\bar{A} = X$.

Lemma 1.6. If A is a subset of B and A is dense, then B is also dense.

Proof. Let $A \subset B$ and let A be dense. By Lemma 1.4 $A \subset B \Rightarrow \bar{A} \subset \bar{B}$. $\bar{A} = X$ by definition of a dense set. $X = \bar{A} \subset \bar{B} \subset X$. Thus $\bar{B} = X$.

Definition 12. A cover of a set A is a collection of sets $\{G_\alpha\}$ such that $A \subseteq \bigcup_\alpha G_\alpha$. If all the sets of a cover are open, we say we have an open cover.

Definition 13. A topological space X is said to be compact if and only if every open cover has a finite subcover. A set K in a topological space X is said to be compact if and only if, with the relative topology, the subspace K is compact; equivalently, a set K in X is compact if and only if every open cover of K by open sets in X has a finite subcover.

<u>Lemma 1.7</u>. Every closed subset of a compact space is compact.

<u>Proof</u>. Let A be a closed set of a compact space X. Let G be a cover of A by open sets in X. Since A is closed, A^c is open. Hence the members of G together with the open set A^c is an open cover of X. Since X is compact, this open cover of X contains a finite subcover of X. In other words, there are a finite number of open sets $O_1, \ldots, O_n$ in G such that $O_1 \cup \ldots \cup O_n \cup A^c = X$. Thus $\{O_1, \ldots, O_n\}$ covers A and A is compact.

CHAPTER II

SOME PROPERTIES OF SPACES WITH NO ISOLATED POINTS

Lemma 2.1. Let X be an infinite set. Then there exist two sets M and N such that the following hold:

1) $M \cup N = X$
2) $M \cap N = \emptyset$ and
3) M and N are each infinite

Proof. Let $\{x_1, x_3, x_5, \ldots\}$ be a countably infinite subset of X. Let $M = \{x_2, x_4, x_6, \ldots\}$ and let $N = X \setminus M$. Since N is the complement of M, we clearly have $M \cap N = \emptyset$ and $M \cup N = X$. Clearly M is infinite, and N is also infinite since it contains the infinite set $\{x_1, x_3, x_5, \ldots\}$.

Definition 14. A point x in a topological space is said to be an isolated point if and only if $\{x\}$ is open.

Lemma 2.2. Let X be a topological space having no isolated points. If S is an infinite subset of X and $x \in S$, then there exists a set M such that the following hold:

1) M is infinite
2) $M \subseteq S$
3) $x \in M$
4) if 0 is open and $x \in 0$, then $0 \nsubseteq M$

Proof. Let S be an infinite subset of X and $x \in S$. Since S is infinite,

$S - \{x\}$ is also infinite. By Lemma 2.1 there exists M_1, M_2 such that $M_1 \cap M_2 = \emptyset$, $M_1 \cup M_2 = S - \{x\}$, and M_1 and M_2 are each infinite.

Case I. All open sets containing x intersect M_1. If U is open and $x \in U$, then $U \cap M_1 \neq \emptyset$. Choose $M = M_2 \cup \{x\}$. Since M_2 is infinite, M is infinite. $M_2 \subseteq S - \{x\} \subseteq S$ and $x \in S$, so $M = M_2 \cup \{x\} \subseteq S$. Hence $M \subseteq S$. Clearly $x \in M$. Now suppose 0 is open and $x \in 0$. By Case I hypothesis 0 intersects M_1. Let $y \in 0 \cap M_1$. Then $y \in M_1$ and $M_1 \subseteq S - \{x\}$ so $y \neq x$. Also since $y \in M_1$ and $M_1 \cap M_2 = \emptyset$, $y \notin M_2$. Thus $y \in M_2 \cup \{x\} = M$. But $y \in 0$, so $0 \not\subseteq M$.

Case II. There exists an open set containing x which does not intersect M_1. Let U be such an open set containing x. Now choose $M = M_1 \cup \{x\}$. Since M_1 is infinite, M is infinite. $M_1 \subseteq S - \{x\} \subseteq S$ and $x \in S$, so $M = M_1 \cup \{x\} \subseteq S$. Clearly $x \in M$. Now suppose 0 is open and $x \in 0$. $x \in 0 \cap U$ and $0 \cap U$ is open, since it is the intersection of two open sets. Since there are no isolated points, $\{x\}$ is not open. Hence $\{x\} \neq 0 \cap U$. But $\{x\} \subseteq 0 \cap U$, so $0 \cap U$ contains a point y different from x. Now $y \in U$, but U does not intersect M_1 by case hypothesis, so $y \notin M_1$. Also since $y \neq x$, we have $y \notin M_1 \cup \{x\} = M$. But $y \in 0$, so $0 \not\subseteq M$. So we have the desired result in all cases.

<u>Definition 15</u>. If S is a subset of the topological space X, then the <u>interior</u> of S (written Int S) is defined to be the union of all open sets which are subsets of S.

<u>Theorem 1</u>. Let X be an infinite topological space having no isolated

points. Then there exists an infinite subset S of X such that Int S $= \emptyset$.

Proof.

A. Choose $x_1 \in X$; this is possible since X is infinite and hence non-empty. Since X is infinite, by Lemma 2.2 there exists a set M_1 such that

1) M_1 is infinite
2) $M_1 \subseteq X$
3) $x_1 \in M_1$
4) if $x_1 \in 0$ and 0 is open then $0 \nsubseteq M$

B. Choose $x_2 \in M_1 - \{x_1\}$; this is possible since M is infinite and hence $M_1 - \{x_1\}$ is non-empty. Since M_1 is infinite, $M_1 - \{x_1\}$ is infinite and thus by Lemma 2.2 there exists a set M_2 such that

1) M_2 is infinite
2) $M_2 \subseteq M_1 - \{x_1\}$
3) $x_2 \in M_2$
4) if $x_2 \in 0$ and 0 is open then $0 \nsubseteq M_2$

C. Now suppose $x_1, M_1, \dots, x_n, M_n$ have been chosen where

1) $M_1, \dots, M_n$ are each infinite
2) $x_i \in M_i$ where $i = 1, \dots, n$
3) $M_i \subseteq M_{i-1} - \{x_{i-1}\}$ where $i = 2, \dots, n$
4) if $x_i \in 0$ and 0 is open then $0 \nsubseteq M_i$ where $i = 1, \dots, n$

D. Choose $x_{n+1} \in M_n - \{x_n\}$; this possible since M_n is infinite and hence $M_n - \{x_n\} \neq \emptyset$. Also since M_n is infinite, $M_n - \{x_n\}$ is infinite. So by Lemma 2.2 there exists a set M_{n+1} such that

1) M_{n+1} is infinite

2) $M_{n+1} \subseteq M_n - \{x\}$

3) $x_{n+1} \in M_{n+1}$

4) if $x_{n+1} \in 0$ and 0 is open then $0 \not\subseteq M_{n+1}$

By mathematical induction we have defined a sequence x_1 , M_1 , x_2, M_2 , ... such that

1) M_i is infinite where $i = 1,2,\ldots$

2) $M_i \subseteq M_{i-1} - \{x_{i-1}\}$ where $i = 2,3,\ldots$

3) $x_i \in M_i$

4) if $x_i \in 0$ and 0 is open then $0 \not\subseteq M_i$ where $i = 1,2,\ldots$

Let $S = \{x_1 , x_2 , x_3 , \ldots\}$. Suppose $i < j$ where i, j are positive integers. $M_j \subseteq M_{j-1} - \{x_{j-1}\} \subseteq M_{j-1} \subseteq M_{j-2} - \{x_{j-2}\} \subseteq M_{j-2} \subseteq \ldots \subseteq M_{i+1} \subseteq M_i - \{x_i\} \subseteq M_i$. Thus $M_j \subseteq M_i$. Also $x_j \in M_j \subseteq M - \{x_i\}$. So $x_i \neq x_j$. Thus $i \neq j \Rightarrow x_i \neq x_j$, so S is infinite. Let 0 be open and $0 \subseteq S$. Suppose $0 \neq \emptyset$. Then let x_j be the element of 0 having smallest subscript. Hence $x_j \in M_j$ and since $x_i \in M_i \subseteq M_j$ for all $i > j$ we see that all other elements of 0 will also be in M_j. Thus $0 \subseteq M_j$. From the induction 0 is open and $x_j \in 0$, so $0 \not\subseteq M_j$. This is a contradiction. Hence it must be that $0 = \emptyset$. Thus the only open sets which lie in S are the empty set. By definition 15 we thus have that Int $S = \emptyset$.

<u>Lemma 2.3</u>. A T_1 space with no isolated points is either empty or infinite.

<u>Proof</u>. Let X be a T_1 space with no isolated points. Since X is T_1 ,

each point is closed and hence each finite subset is closed being the union of a finite number of closed sets. If $X \neq \emptyset$ and X is finite, then let $x \in X$. $X - \{x\}$ is finite and hence closed. Therefore $\{x\}$ is open so x is an isolated point. This is a contradiction. So either X is empty or finite.

CHAPTER III

MI-SPACES

The main results of this paper are embodied in this chapter. I will show that a group of spaces called MI-Spaces has the property that each compact set is finite. The term pseudo-finite will be used to designate this property.

Definition 16. A topological space X is said to be an MI-space if it possesses no isolated points and each dense set is open.

Lemma 3.1. An MI-space is a T_1 space.

Proof. Let X be an MI-space and let $x \in X$. $\{x\}$ is not open since X is an MI-space. Thus every open set containing x must contain a point other than x. Hence every open set containing x contains a point of $\{x\}^c$. Therefore $x \in \overline{\{x\}^c}$. Since $\{x\}^c \subseteq \overline{\{x\}^c}$ we have $X = \{x\} \cup \{x\}^c \subseteq \overline{\{x\}^c}$. Thus $\{x\}^c$ is dense. By definition of an MI-space each dense set is open, so $\{x\}^c$ is open. Therefore $\{x\}$ is closed. Therefore X is a T_1 space.

Lemma 3.2. The empty set is an MI-space.

Proof. The empty set has no points; hence no isolated points. The only subset of $\emptyset$ is $\emptyset$ itself. $\emptyset$ is open, hence each dense set is open. Thus $\emptyset$ is an MI-space by Definition 16.

Lemma 3.3. In an MI-space each set with empty interior has a dense complement.

Proof. Let X be an MI-space and let $A \subseteq X$ such that Int $A = \emptyset$. Let $y \in X$. Let 0 be an open set such that $y \in 0$; 0 cannot lie in A since $0 \neq \emptyset$ and Int $A = \emptyset$. Therefore $0 \not\subseteq A$. Hence $0 \cap A^c \neq \emptyset$. Since 0 is arbitrary, every open set containing y contains a point of A^c. Therefore y is a point of closure of A^c; i.e., $y \in \overline{A^c}$. Hence $X \subseteq \overline{A^c}$. Clearly $\overline{A^c} \subseteq X$, so $\overline{A^c} = X$. By definition A^c is dense.

Definition 17. A set A of a topological space X is totally isolated if and only if it has no limit points whatsoever.

Theorem 2. In an MI-space each set with empty interior is totally isolated.

Proof. Let X be an MI-space. Let $A \subseteq X$ such that Int $A = \emptyset$. Suppose A has a limit point x.

Case I. $x \in A$. I will show $A^c \cup \{x\}$ is a non-open dense subset of X. First I will show $A^c \cup \{x\}$ is not open. Since x is a limit point of A, every open set containing x contains a point of A different from x. But no open set lying in $A^c \cup \{x\}$ contains a point of A different from x, so no open set containing x lies in $A^c \cup \{x\}$. Hence $A^c \cup \{x\}$ is not open. A is dense by Lemma 3.3. Since $A^c \subseteq A^c \cup \{x\}$, $A^c \cup \{x\}$ is dense by Lemma 1.6. Thus it has been shown that $A^c \cup \{x\}$ is a non-open dense subset of X. This contradicts the definition of X being and MI-space.

Case II. $x \notin A$; i.e., $x \in A^c$. I will show that A itself is a non-open dense subset of X. Since x is a limit point of A, every open set containing x contains a point of A different from x. In particular, every open set containing x contains a point of A; i.e., a point not is A^c. Hence no open set containing x can lie in A^c. Therefore A^c is not open. A^c is dense by Lemma 3.3. So again we have a contradiction. So our original assumption that A has a limit point is false. Therefore $A' = \emptyset$. So by Definition 17 we have A is totally isolated.

<u>Theorem 3</u>. In an MI-space each totally isolated set has empty interior.

<u>Proof</u>. Let X be an MI-space and let A be a totally isolated subset of X such that $x \in \text{Int } A$. There is an open set 0 in X such that $x \in 0 \subseteq X$. Since A is totally isolated, each point $y \in A$ is such that $\{y\}$ is open relative to A. $x \in A$, so $\{x\}$ is open relative to A. There is an open set U in X such that $U \cap A = \{x\}$. Since $0 \subseteq A$, $U \cap 0 \subseteq U \cap A = \{x\}$. Therefore $U \cap 0 = \{x\}$. Hence $\{x\}$ is open. This gives us a contradiction since X contains no isolated points. That is, no point such that $\{x\}$ is open.

<u>Theorem 4</u>. A subspace of an MI-space is also an MI-space provided it has no isolated points.

<u>Proof</u>. Let X be an MI-space and let K be a subspace of X with no isolated points. Let $A \subseteq K$ such that A is dense in K. Since $A \subseteq A \cup K^c$ then $A \cup K^c$ is dense in X by Lemma 1.6. By definition of an MI-space each dense set is open. Thus $A \cup K^c$ is open in X. $(A \cup K^c) \cap K = A$ since $A \subseteq K$. Hence A is an open set in K. Thus each dense set in K is open in K. By hypothesis K has no isolated points so K is an MI-space.

Theorem 5. If X is an MI-space and $X \neq \emptyset$, then X is infinite.

Proof. Let $A \subseteq X$. Then A is finite and $A = \bigcup \{\{x\} \mid x \in A\}$, a union of a finite set of closed sets. So A is closed. Hence each subset of X is closed and thus each subset of X is open, so X has the discrete topology. Since $X \neq \emptyset$, let $x \in X$. Then $\{x\}$ is open, so x is an isolated point; a contradiction, since X is an MI-space. Hence there are no finite non-empty MI-spaces.

Theorem 6. Each non-empty MI-space is non compact.

Proof. Let X be an MI-space which is not empty. By Theorem 5 X is infinite. Let A be an infinite subset of X with Int $A = \emptyset$. The existence of such a set follows from Theorem 1. In an MI-space each set with empty interior is totally isolated by Theorem 2. Hence A is an infinite totally isolated subset of X. Let $x \in A$, then $\{x\}$ is open relative to A, since A is totally isolated. $A = \bigcup \{\{x\} \mid x \in A\}$. The class $\mathcal{A} = \{\{x\} \mid x \in A\}$ is a cover of A and is an open cover of A since all subsets of a discrete space are open. A is closed and discrete by Lemma 1.5. No proper subclass of $\mathcal{A}$ is a cover of A, for if we take out one $\{x\}$ we would no longer have a cover of A. Hence $\mathcal{A}$ is infinite since A is infinite. Therefore the open cover $\mathcal{A}$ of A contains no finite subcover. Hence A is not compact. Since A is closed, X is not compact by Theorem 1.7.

Theorem 7. Each MI-space is pseudo-finite.

Proof. Let K be a compact subset of the MI-space X. Int $K \subseteq K$ and Int K is open in K since it is the intersection of an open set, Int K, with K. So K - Int K is closed in K since its complement, Int K, is open in K. Hence K - Int K is a closed subset of the compact set K. By Lemma 1.7 K - Int K is compact.

Suppose $x \in$ Int (K - Int K). Then there exists an open set 0 such that $x \in 0 \subset K$ - Int K. Clearly $0 \subset K$, so $x \in$ Int K and $x \notin$ K - Int K. But $x \in$ Int (K - Int K) $\subset$ K - Int K. This is a contradiction, so Int (K - Int K) is empty. So K - Int K is totally isolated by Theorem 2.

By Lemma 1.5 K - Int K has the discrete topology. Let $C = \{\{x\} \mid x \in (K - \text{Int } K)\}$. This is an open cover of K - Int K since the relative topology is discrete. The only subcover of C is C itself. If K - Int K is infinite, then there is no finite subcover of K - Int K, so it would not be compact. Since it is compact, then K - Int K is finite.

Let I be the set of isolated points of K; i.e., $I = \{x \mid x \in K$ and $\{x\}$ open relative to $K\}$. Let $x \in I$. Suppose $x \in$ Int K. Since $x \in I$, there exists an open set 0 such that $0 \cap K = \{x\}$. So $0 \cap$ Int $K = \{x\}$. Hence $\{x\}$ is open in X. This is a contradiction since X is an MI-space and thus has no isolated points. Therefore $x \notin$ Int K. But $I \subseteq K$ and $x \in I$ so $x \in$ (K - Int K). Thus $I \subseteq$ (k - Int K). I is finite since it is a subset of a finite set. $I = \bigcup \{\{x\} \mid x \in I\}$, a union of open sets in K. Hence I is open in K. K - I is closed in K so again by Lemma 1.7 K - I is compact. Hence K - I is a compact subset of X with no isolated points. Hence K - I is an MI-space by Theorem 3. By Theorem 4 K - I is empty since each non-empty MI-space is non-compact. Since $K - I = \emptyset$, then $K \subseteq I$. Clearly as before $I \subseteq K$. Therefore K = I. Hence K is finite since I is finite. Thus each compact subset K is finite. Hence the whole space X is pseudo-finite.

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